

# Mass as a relativistic quantum observable

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A field state containing photons propagating in different directions has a non vanishing mass which is a quantum observable. We interpret the shift of this mass under transformations to accelerated frames as defining space-time observables canonically conjugated to energy-momentum observables. Shifts of quantum observables differ from the predictions of classical relativity theory in the presence of a non vanishing spin. In particular, quantum redshift of energy-momentum is affected by spin. Shifts of position and energy-momentum observables however obey simple universal rules derived from invariance of canonical commutators.

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Since the birth of classical mechanics, the two fundamental concepts of space and mass have essentially been regarded as physical notions defined simultaneously but independently of one another. While space is the arena where motion takes place, mass is a measure of inertia associated with a given moving object. General theory of relativity introduces a dynamical connection between space and mass since inertia and gravitation are interpreted in terms of geometrical properties of space-time [1]. We will argue in the present letter that mass and space-time have also to be considered as closely connected in quantum physics, even in the absence of gravitation. When set in the framework of quantum theory, seminal arguments developed by Einstein in the early years of relativity theory indeed entail that mass should not be considered as a constant, but rather as a relativistic quantum observable the properties of which allow to define positions in space-time.

In accordance with the standard relativistic prescription [2], the mass  $M$  associated with a physical system will be defined as the Lorentz invariant built on the energy-momentum variables  $P_\mu$

$$M = \sqrt{\eta^{\mu\nu} P_\mu P_\nu} = \sqrt{P_\mu P^\mu} \quad (1)$$

Throughout the letter, the Minkowski tensor  $\eta^{\mu\nu}$  is used to raise or lower indices. Definition (1) leads to attribute inertia to any kind of energy and implies that mass is a dynamical quantity rather than a constant parameter. For instance, the mass of an atom changes when it emits or absorbs a photon. It is also known that energy and momentum are shifted under transformations to accelerated frames. The proportionality of these shifts versus acceleration and position measured along the direction of acceleration reveals the equivalence between uniform acceleration and constant gravity [3]. Mass, as defined by equation (1), also undergoes a redshift (or a blueshift) under transformations to accelerated frames.

These issues raise intriguing questions in the context of quantum theory. Since energy and momentum are quantum observables, redshifts have to be written in terms of quantum positions which necessarily differ from parameters used to map space-time. Hence, relativistic transformations of physical observables, like the Einstein redshift law, may no longer be given their usual derivation from classical relativity theory. Moreover, covariance properties are associated with conventional choices of coordinate maps, and the universal form of the redshift laws must be given a new interpretation in terms of quantum observables. Dimensional relations between mass and time connect this question to the similar issue of universality of relativistic transformations of space-time observables. The velocity of light  $c$  and the Planck constant  $\hbar$  are fundamental constants, so that mass scale is expected to vary as the inverse of time scale under frame transformations [4]. Clearly, any formulation where the status of mass is that of a constant classical parameter does not have the ability to solve these questions. A few attempts have been made to develop theoretical approaches where mass is allowed to vary or described as a quantum variable but they rely on specific features which are added to the standard formalism and do not appear to be universally admitted [5].

The approach developed in the present letter uses the fact that, although photons are massless, any field state containing photons propagating in at least two different directions corresponds to a non vanishing mass (1). As an illustration, one may consider a state built on two counterpropagating photons which corresponds to a vanishing momentum and therefore has a non vanishing mass equal to its energy. We will thus be able to study a mass treated as a quantum observable in the simple framework of free electromagnetic theory. We will then show that the mass shift under transformations to accelerated frames may be written in terms of space-time observables canonically conjugated to energy-momentum observables. Symmetry of electromagnetism under Lorentz transformations plays a prime role

in the deduction of relativistic effects. Here, we will use conformal invariance of electromagnetism which allows to deal with uniformly accelerated frames in classical [6] as well as quantum physics [7].

Conformal symmetry is basically described by conformal algebra, that is the set of commutators of its generators. Commutators of generators  $[\Delta_a, \Delta_b]$ , and more generally of all observables, will be written under the form

$$(\Delta_a, \Delta_b) \equiv \frac{1}{i\hbar} [\Delta_a, \Delta_b] \quad (2)$$

In the following, we will often use the Jacobi identity

$$((\Delta_a, \Delta_b), \Delta_c) = (\Delta_a, (\Delta_b, \Delta_c)) - (\Delta_b, (\Delta_a, \Delta_c)) \quad (3)$$

Conformal generators are integrals of the energy-momentum density of the quantum fields which are preserved under field propagation [8]. They are defined in such a manner that they vanish in vacuum, in consistency with conformal invariance of vacuum [9]. Corresponding respectively to translations (energy-momentum  $P_\nu$ ), rotations (angular momentum and Lorentz boosts  $J_{\nu\rho}$ ), dilatation ( $D$ ) and conformal transformations to uniformly accelerated frames ( $C_\nu$ ), the generators obey the following commutation relations

$$\begin{aligned} (P_\mu, P_\nu) &= 0 & (J_{\mu\nu}, P_\rho) &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ (J_{\mu\nu}, J_{\rho\sigma}) &= \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} \\ (D, P_\mu) &= P_\mu & (D, J_{\mu\nu}) &= 0 \\ (P_\mu, C_\nu) &= -2\eta_{\mu\nu} D - 2J_{\mu\nu} \\ (J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho} C_\mu - \eta_{\mu\rho} C_\nu \\ (D, C_\mu) &= -C_\mu & (C_\mu, C_\nu) &= 0 \end{aligned} \quad (4)$$

Equations (4) follow from the commutation rules of quantum fields. They describe the commutators of the generators considered as quantum observables as well as the frame transformations of the generators considered as relativistic observables. The  $P_\mu$ 's are commuting components of a Lorentz vector. The  $C_\mu$ 's obey the same property. The conformal weights, determined by commutation relations with  $D$ , are opposite for  $C_\mu$ 's and  $P_\mu$ 's. The  $J_{\mu\nu}$ 's form a Lorentz tensor with a vanishing conformal weight. Finally, the commutators between the  $P_\mu$ 's and  $C_\nu$ 's describe the shifts of energy-momentum under transformations to accelerated frames. As quantum analogs of redshift laws, they play an important role in the following discussions.

Focusing our interest on the mass observable  $M$  defined in equation (1), we write its shifts under different frame transformations

$$\begin{aligned} (P_\mu, M) &= (J_{\mu\nu}, M) = 0 & (D, M) &= M \\ (C_\mu, M) &= 2(\eta_{\mu\rho} D - J_{\mu\rho}) \cdot \frac{P^\rho}{M} \end{aligned} \quad (5)$$

We have taken care of non commutativity of observables by introducing a symmetrised product represented by the “ $\cdot$ ” symbol. As expected, mass shift vanishes for Poincaré transformations and is proportional to  $M$  for dilatation. For transformations to accelerated frames, mass shift exhibits a state dependence that allows to define space-time observables. To emphasize this key point, we consider the general transformation to an accelerated frame

$$\Delta = \frac{a^\mu}{2} C_\mu \quad (6)$$

where the classical numbers  $a^\mu$  represent accelerations along the various space-time directions. The mass shift  $(\Delta, M)$  corresponding to such a transformation may be written

$$(\Delta, M) = a^\mu M \cdot X_\mu \quad X_\mu \equiv (\eta_{\mu\rho} D - J_{\mu\rho}) \cdot \frac{P^\rho}{M^2} \quad (7)$$

Shifts under Poincaré transformations and dilatation of the observables  $X_\mu$  defined in this manner are derived from Jacobi identity (3) and conformal algebra (4)

$$(P_\mu, X_\nu) = -\eta_{\mu\nu} \quad (D, X_\mu) = -X_\mu \quad (J_{\mu\nu}, X_\rho) = \eta_{\nu\rho} X_\mu - \eta_{\mu\rho} X_\nu \quad (8)$$

They are identical to the shifts of coordinate parameters under corresponding map changes. The observables  $X_\mu$  are conserved quantities since they have been built on conformal generators. These remarks suggest that these quantum

observables have to be interpreted as the positions of some event in space-time. In the specific case of a state consisting in two light pulses originating from a given point [10], observables  $X_\mu$  may effectively be identified with the space-time positions of the coincidence event. The same interpretation still holds in the general case of an arbitrary number of photons propagating along any direction, provided the position of the coincidence event is understood as an average over the energy-momentum distribution associated with the quantum field state. In the spirit of the discussion presented in the introduction, relations (7) reveal an intimate connection between mass and space-time. The mass shift under frame transformation (6) is indeed proportional to the mass  $M$  itself, to the acceleration  $a^\mu$  and to the position  $X_\mu$  measured along the direction of acceleration. This is exactly the form expected for the potential energy of a mass in a constant gravitational field so that equation (7) may be interpreted as the redshift of mass written in terms of quantum positions. This law hence constitutes a statement of equivalence between acceleration and gravity, valid in the quantum domain.

The first of equations (8) also means that observables  $X_\mu$  are conjugated to energy-momentum operators. Canonical commutation relations are thus embodied in conformal algebra (4). Although they have been derived directly from standard quantum formalism, these equations conflict statements which are often claimed to be unavoidable consequences of this formalism [11]. As a matter of fact, equations (7,8) contain the definition of a time operator  $X_0$  besides that of space operators. Furthermore, this operator enters an energy-time canonical commutation relation which has the same form as momentum-space relations while the whole set of relations satisfies Lorentz invariance. In accordance with the definition of conformal generators (4), time as well as space operators are defined only for field states orthogonal to vacuum, so that difficulties related to hermiticity in the definition of phases are bypassed [12]. We may stress again that the time operator  $X_0$  is a localisation observable, that is precisely the date associated with an event, and therefore differs from any kind of evolution parameter [10]. The shifts studied in the present letter do not represent an effect of evolution but rather an effect of frame transformations. In particular, canonical commutators describe the shifts of space-time observables under space-time translations. With these precisions kept in mind, relations (7,8) contain a quantum definition for a time operator [13].

The previous derivations still hold in the presence of spin which is however known to play a crucial role in the problem of localisability of quantum fields [14]. This is clearly illustrated by the evaluation of commutators of space-time components, using Jacobi identity (3), conformal algebra (4) and canonical commutators (8)

$$\begin{aligned} (M \cdot X_\mu, M \cdot X_\nu) &= \frac{1}{4} ((C_\mu, M), (C_\nu, M)) = J_{\mu\nu} \\ M^2 \cdot (X_\mu, X_\nu) &= J_{\mu\nu} - (P_\mu \cdot X_\nu - P_\nu \cdot X_\mu) \equiv S_{\mu\nu} \end{aligned} \quad (9)$$

These commutators are given by expressions  $S_{\mu\nu}$  defined as the difference between angular momentum  $J_{\mu\nu}$  and orbital angular momentum built on energy-momentum and space-time observables. This internal angular momentum is directly related to the Pauli-Lubanski vector  $S^\mu$  which is the standard covariant generalisation of spin in relativistic kinematics [8,15]

$$S_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} S^\rho \frac{P_\sigma}{M} \quad S^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} \frac{P_\sigma}{M} \quad (10)$$

where  $\epsilon_{\mu\nu\lambda\rho}$  is the completely antisymmetric Lorentz tensor of rank 4.

Equations (9) entail that commuting positions may be defined only in the specific case of a vanishing spin. Since spin is a basic ingredient of quantum physics, the preceding relation shows that notions inherited from classical differential geometry cannot be applied without modification to quantum observables. As a further illustration, the redshift  $(\Delta, P_\nu)$  of energy-momentum under the frame transformation (6) is evaluated from (4) as

$$(\Delta, P_\nu) = a^\mu (\eta_{\mu\nu} D - P_\mu \cdot X_\nu + P_\nu \cdot X_\mu - S_{\mu\nu}) \quad D = P_\rho \cdot X^\rho \quad (11)$$

The expression of  $D$  which appears in equation (11) is a direct consequence of the definition (7) of position observables. The quantum redshift law (11) differs from the classical one, since the redshift of energy-momentum now depends not only on positions but also on spin observables. Einstein's law must be regarded as a classical approximation valid in the limiting case where spin contributions are negligible. In all cases however, mass shift keeps a classical form (7) which is not affected by spin contributions. This is related to the fact that these contributions are orthogonal to energy-momentum in equation (11).

The redshift law (11) has a universal form dictated by conformal algebra, although this form differs from the classical one. Shifts of positions  $X_\mu$  take the form predicted by classical relativity for Poincaré transformations and dilatation (see (8)) but not for transformations to accelerated frames which, already for a vanishing spin, mix positions

with energy-momentum [10,16]. The derivation of these shifts in the presence of spin lies outside the scope of the present letter. We may however give simple expressions which provide interesting insights concerning the question of universality of relativistic transformations. To this aim, we note that the canonical commutators  $(P_\mu, X_\nu)$  are pure numbers which are invariant under all frame transformations. Invariance of canonical commutators corresponds in fact to the statement that the Planck constant is constant like the velocity of light [4], not only for Poincaré transformations but also for conformal transformations to accelerated frames. Since  $(P_\mu, X_\nu)$  commutes with the generator  $\Delta$  of such transformations, Jacobi identity (3) leads to the following relation

$$((\Delta, X_\nu), P_\mu) = ((\Delta, P_\mu), X_\nu) \quad (12)$$

To discuss the significance of this relation, we first remind that  $(\Delta, X_\nu)$  is the shift of position, so that  $((\Delta, X_\nu), P_\mu)$  is the variation of this shift under a translation. This expression is thus a quantum analog of the matrix  $\frac{\partial \delta x^\nu}{\partial x^\mu}$  which is used in classical differential geometry to describe the change of a vector under the infinitesimal coordinate deformation  $\delta x^\nu(x^\mu)$ . The second double commutator  $((\Delta, P_\mu), X_\nu)$  appearing in (12) is similar to the first one with the roles of position and energy-momentum interchanged. The presence of a non vanishing spin affects the expression (11) of the redshift  $(\Delta, P_\mu)$  as well as the evaluation of commutators with position components (see (9)). In this context, equation (12) exhibits a non trivial relation between shifts of quantum observables associated with position and energy-momentum which is still valid in the presence of a non vanishing spin. Furthermore, an explicit evaluation of the double commutators shows that the expressions appearing in equation (12) have a classical form which is not affected by spin

$$((\Delta, X_\nu), P_\mu) = ((\Delta, P_\mu), X_\nu) = -\eta_{\mu\nu} a^\rho X_\rho - a_\mu X_\nu + a_\nu X_\mu \quad (13)$$

In the particular case  $\mu = \nu = 0$ , the two expressions connected by equation (12) represent respectively the shift of a clock rate under acceleration and the commutator with time of the redshift of energy. Equation (13) identifies their common expression in terms of quantum positions with the prediction of classical relativity although the time shift  $(\Delta, X_0)$  and the redshift  $(\Delta, P_0)$  both differ from classical predictions.

To sum up our results, we have defined a mass observable for a quantum field state and derived a position in space-time from the redshift of this mass, in accordance with Einstein redshift law. Space-time observables defined in such a manner have been found to be canonically conjugated to energy-momentum observables. We have also derived transformation laws which differ from the predictions of classical relativity theory but keep a universal form dictated by conformal algebra. Commutators of position components as well as redshifts of energy-momentum are affected by spin. Covariance rules which reflect the universality of predictions of classical relativity theory have been reformulated as statements of invariance of canonical commutators under frame transformations.

Strictly speaking, all these results have been established only for specific quantum systems, namely free electromagnetic fields with photons propagating in at least two different directions. Arguing that their range of interest is restricted to such specific systems would however lead to considerable difficulties concerning the consistency between relativistic and quantum properties as well as the known universality of these properties. Since a consistent theory including general relativity and quantum theory is still lacking, it is worth discussing basic concepts at the interplay of quantum theory, inertia and gravity. The results of the present letter plead for a theoretical frame where mass and space-time positions would be treated as relativistic quantum observables with their basic properties embodied in symmetries rather than in classical covariance rules [17]. They already demonstrate that equivalence between uniform acceleration and constant gravity fits perfectly well in such a frame. Reanalysing the questions raised by gravitational physics should probably lead to reconsider the role played in quantum theory by basic geometrical concepts [18].

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